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STABILIZATION OF A QUASI-CONSERVATIVE SYSTEM SUBJECTED TO HIGH FREQUENCY EXCITATION†

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The existence and stability conditions for the steady motions and equilibrium positions of non-linear quasi-conservative systems with fast external perturbations having quasi-periodic and random components are investigated. A change of variables is proposed which reduces Lagrange's equations of the system to standard form. It is shown the averaged system of the first approximation has a canonical form and the effect of fast perturbations (not necessarily potential) is equivalent to a change in the system's potential. This leads to stabilization of unstable equilibrium positions and to the appearance of additional stationary points different from the equilibrium positions of the unperturbed system. The approach used is illustrated by examples; the stability of a pendulum on an elastic suspension when there is suspension point, and the steady motion of a sphere subjected to a high-frequency load. The critical loading of a double pendulum loaded by a pulsating tracking force is estimated. A form of wide-band random perturbations capable of stabilizing the system is considered.

Stabilization of the unstable equilibrium positions of dynamical systems in the case of high-frequency or quasi-periodic perturbations of their parameters has been known for years [1–5]. The stabilization of non-linear systems with random perturbations has mostly been studied for systems with one degree of freedom 6, 7. The question of the form of wideband action capable of stabilizing the system has been discussed [8].

The main attention has been given to the stabilization of an unstable equilibrium position of an unperturbed system. However, it follows from general results [3, 4] that high-frequency parametric perturbations may lead to the assurance of additional stable positions of relative equilibrium not found in unperturbed systems. This phenomenon has been studied in detail for mechanical systems such as a pendulum with a non-linear suspension [4], a spherical pendulum [9–11] and a Lagrange top [12] subjected to kinematic excitation.

Methods of investigating perturbed systems have been developed [3, 4] in which the equations of motion are reduced to standard form and the stationary points of the averaged equations of the first approximation are analysed. The direct use of this approach to analyse complex mechanical systems, represented by Lagrange's equations, requires preliminary transformations of the equations of motion and does not enable an explicit relation between the stability conditions and the system's structure to be obtained.

In this paper we introduce a change of variables which reduces Lagrange's equations of the perturbed system to standard form, allowing of averaging. It is shown that the effect of fast perturbations in the first approximation reduces to a modification of the structure of the potential forces. The deformation of the potential of the system can give rise to new stable stationary points, different from the stable equilibrium positions of the unperturbed system. This relation between the stability and the change in potential was also obvious in each of the special cases investigated previously in [4, 9–12]. In this paper additional potential forces are found in explicit form for a fairly wide range of systems. It is shown that the effect of a change in potential also holds for both deterministic and random non-potential perturbation.

1. INITIAL PROPOSITIONS AND FORMULATION OF THE VIBRATIONAL STABILIZATION PROBLEM

Suppose τ is the "slow time" associated with the system, $x(\tau)$ and $x' = dx/d\tau$ are n-dimensional vectors of the generalized coordinates and velocities, and $T(x, x')$ is the kinetic energy of the system, specified by a matrix $A(x)$. The matrix $A(x)$ is positive – definite in the domain of variation of the variables considered. The potential energy of the system is denoted by the function $\Phi(x)$ and the vector of potential

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forces is denoted by $Q(x) = d\Phi/dx$. The vector of generalized forces, in general, is non-potential and can be written, in the general case, in the form $S_\varepsilon(\tau, x) = \varepsilon^{-1}S(\tau/\varepsilon, x)$, where ε is a small parameter, and does not include a constant component. We suppose that $S_\varepsilon(\tau, x)$ is a zero mean function of τ. Hence it follows that the system's motion is considered as a slow process compared to the perturbations. A similar separation of the fast and slow motions is typical, for example, for pendulum-like systems with a kinematic excitation, where the generalized force $S_\varepsilon(\tau, x)$ corresponds to inertia forces generated by rapid vibration of the suspension point [4–12].

We will consider the "fast time" $t = \tau/\varepsilon$ as the independent variable and write the equations of motion in the form of Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial x^{\cdot}} - \frac{\partial T}{\partial x} + \varepsilon^2 Q(x) = \varepsilon S(t, x) \tag{1.1}$$

where

$$x^{\cdot} = dx/dt, T(x, x^{\cdot}) = [(x^{\cdot})^T A(x) x^{\cdot}]/2$$

The functions $A(x)$, $Q(x)$ and $S(t, x)$ are assumed to be sufficiently smooth and allow of the necessary transformations.

We introduce the new variables

$$x = q, p = A(q)q^{\cdot} \tag{1.2}$$

which yields

$$q^{\cdot}(p, q) = v(p, q) = a(q)p, \quad a(q) = A^{-1}(q) \tag{1.3}$$

We define the total energy of the system as

$$E(p, q, \varepsilon) = G(p, q) + \varepsilon^2\Phi(q); \quad G(p, q) = T(q, v(p, q)) \tag{1.4}$$

where, by (1.1)–(1.3), $G(p, q) = p^T a(q)p/2$. The equations of motions for the variables q and p can then be written in the form

$$q^{\cdot} = \frac{\partial E}{\partial p}, \quad p^{\cdot} = -\frac{\partial E}{\partial q} + \varepsilon S(t, q) \tag{1.5}$$

Since the motion is slow, we introduce a new slow variable y by the formula

$$p = \varepsilon y + \varepsilon V(t, q); \quad S(t, q) = \partial V(t, q)/\partial t \tag{1.6}$$

where $V(t, q)$ does not contain a constant component. This assumption enables as to construct a unique function $V(t, q)$ corresponding to the given function $S(t, q)$. Unlike the replacement of variables for second-order equations [4], transformation (1.6) does not require inversion of matrices and results in equations in standard form, in which only the first power of ε occurs on the right-hand sides

$$\begin{aligned} q^{\cdot} &= \varepsilon a(q)y + \varepsilon F_1(t, q) \\ y^{\cdot} &= -\varepsilon\frac{\partial}{\partial q}\{[\Phi(q) + F(q)] + \frac{1}{2}[y^T a(q)y]\} + \varepsilon F_2(t, q, y) \end{aligned} \tag{1.7}$$

where

$$F(q) = \langle F(t, q)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{M}\, F(t, q)\,dt, \quad F(t, q) = \frac{1}{2}V^T(t, q)a(q)V(t, q) \tag{1.8}$$

Here and below M is the mathematical expectation operator.

Transformation (1.6) enables us to define the terms $F_1(t, q)$ and $F_2(t, q)$ in such a way that the average $\langle F_{1,2}(t, q, y)\rangle = 0$. The coefficients of (1.7) are assumed to be sufficiently smooth to allow necessary transformation. Restrictions are associated with the process $F(t, q)$. Further we suppose that the function

$F(t, q)$ is taken as a sum of bounded quasi-periodic and random processes, and the averaging principle [13] (or the stochastic averaging principle [14]) is applicable to system (1.7).

If $S(t, q) = v^{\cdot}(t)Y(q)$, where $v^{\cdot}(t)$ is a scalar, the coefficient $F(t, q)$ takes a simpler form

$$F(t,q) = \frac{1}{2}v^2 f(q), \quad F(q) = \frac{1}{2}\sigma^2 f(q)$$

$$f(q) = Y^T(q)a(q)Y(q), \quad \sigma^2 = \langle v^2(t)\rangle \tag{1.9}$$

Together with Eq. (1.7) we consider the truncated system

$$q_0^{\cdot} = \varepsilon a(q_0)y_0, \quad y_0^{\cdot} = -\varepsilon\frac{\partial}{\partial q_0}\left\{[\Phi(q_0) + F(q_0)] + \frac{1}{2}[y_0^T a(q_0)y_0]\right\} \tag{1.10}$$

Relations (1.3) and (1.4) imply that Eqs (1.10) describe the motion of a conservative system with Hamiltonian $H(y_0, q_0) = G(y_0, q_0) + U(q_0)$, where q_0 and y_0 are the vectors of the generalized coordinates and momenta, $G = y_0^T a(q_0)y_0/2$ is the kinetic energy of the system and $U(q_0) = \Phi(q_0) + F(q_0)$ is the potential energy. We will call the function $U(q_0)$ "the effective potential". The term $F(q_0)$ corresponds to the contribution of fast perturbations to the effective potential and the function $K(q_0) = \partial F(q_0)/\partial q_0$ defines the additional potential forces. Hence the effect of fast perturbations in the first approximation reduces to a change in the structure of the potential forces compared with the unperturbed system.

Standard procedures [15] can be employed to study the stationary solutions of Hamilton system (1.10). If system (1.10) does not contain cyclic coordinates, the stationary points q^* of system (1.10) can be found as the extremal points of the potential $U(q)$, i.e. from the equations

$$\partial U/\partial q = R(q) = Q(q) + K(q) = 0, q \in R_n \tag{1.11}$$

The point q^* corresponding to a strict minimum of $U(q_0)$ is a stable stationary point of truncated system (1.10).

Suppose the coefficients $a(q)$, $\Phi(q)$, $F(q)$ depend only on the coordinates q_i, where $i = 1, m$. Then system (1.10) allows of separation of the cyclic coordinates so the system's dimensionality is diminished. The truncated system for the positional coordinates retains the form (1.10), but the function $\Phi(q)$ is interpreted as the corresponding Routh potential [15]. Steady solutions can be obtained from Eq. (1.11) as extrema of the corresponding effective Routh potential.

It follows from the properties of a Hamilton system [15] that, in the linear approximation a stationary solution can be either unstable or non-asymptotically stable. In the first case the original system (1.7) is also unstable (with probability 1 in the case of random perturbations). In the second case the solution of the perturbed system remains in a small neighbourhood of the equilibrium position at least in the time interval $t \sim 1/\varepsilon$. An analysis of the stability requires a consideration of a higher-order asymptotic expansion [13, 16].

If the equations of motion contain dissipative forces, the corresponding stationary solution can become asymptotically exponentially stable in the linear approximation. For uniformly bounded periodic or almost periodic perturbations, the stationary solution of the perturbed system is also asymptotically stable and remains within an ε-neighbourhood of the stationary point [13]. If the perturbations are random stationary processes, which cannot be regarded as bounded, then, under previous assumptions, system (1.7) is asymptotically stable in probability [14], since almost all trajectories converge to a small vicinity of the stationary point as $t \to \infty$, and large deviations (of order 1) from this point occur with probability $\sim \exp(-C/\varepsilon)$ [14].

Analysis of possible stationary solutions. In the general case of non-linear systems the stationary points found from Eq. (1.11) do not coincide with the equilibrium positions of the unperturbed system. Consider some special cases.

Suppose the kinetic energy matrix A is independent of x and the perturbation occur additively, that is $S(t, x) \equiv S(t)$. Then $F(t, q) \equiv 0$, $K(q) \equiv 0$, that is, an additive fast perturbation does not change the equilibrium positions of a non-linear system and does not affect their stability.

Suppose the unperturbed system does not contain cyclic coordinates. From Eq. (1.11) it then follows that the unperturbed equilibrium positions persist if the roots of the equation $K(q) = 0$ are identical with the roots of the equation $Q(q) = 0$. However, the conditions for these equilibrium positions to be stable may change.

Consider the example of a linear system. The equations of motion have the form

$$x^{\cdot\cdot}+\varepsilon^2 Cx+\varepsilon S(t)x=0,\quad S(t)=V^{\cdot}(t) \tag{1.12}$$

If the matrix C is positive-definite, the unperturbed system when $S(t) = 0$ has the stable equilibrium $x = 0$.

Truncated system (1.10) reduces to the form

$$q_0^{\cdot}=\varepsilon y_0,\quad y_0^{\cdot}=-(C+D)q_0 \tag{1.13}$$

where, by (1.8), D is a matrix with components $D_{ij} = \langle V_i(t)V_j(t)\rangle$. If the matrix $C + D$ is non-degenerate, system (1.13) retains the equilibrium position $q_0 = 0$. Its stability is determined by the properties of the roots of the characteristic equations

$$\det|Ip^2+(C+D)|=0 \tag{1.14}$$

For a correct choice of the perturbation intensity, the matrix $C + D$ becomes positive – definite, even if the matrix C does not possess this property. An unstable equilibrium position can thus be stabilized by a fast parametric perturbation [3].

A random perturbation. If the perturbation is periodic or quasi-periodic and its frequencies are much higher than the natural frequencies of the system, the perturbation can be regarded as a rapidly varying process compared to the motion of the system. In this case a small parameter is introduced as the ratio of the frequencies. If the perturbation is a random stationary process with a continuous spectrum, the concept of a rapidly varying perturbation and the introduction of a small parameter require an explanation.

Suppose we represent the vector of generalized forces in the form $S(t, x) = s(t)Y(x)$, where $s(t) = v^{\cdot}(t)$ is a scalar stationary Gaussian process with continuous spectral density in the form of a rational fractional function [17]. We now find the conditions under which a with a continuous spectrum can be considered as a high-frequency process and all previous transformations remain valid. It follows from (r, t) and (r, s) that all transformations are valid if the process $s(t)$ is integrable, and its spectrum can be represented in the form

$$R_k(\omega)=\omega^{2k}\,|\,P_{2n}(i\omega)\,||\,L_{2m}(i\omega)\,|^{-1} \tag{1.15}$$

where $k = 1$, and $L_{2m}(p)$ and $P_{2n}(p)$ are polynomials of degrees $2m$ and $2n$, respectively, where $2m \geqslant 2(n + k)$. Hence $R(0) = 0$, and the spectral density $R(\omega)$ is small for fairly small $\omega < \omega^*$. By analogy with the well-known definition [17], the frequency ω^* is called the cut-off frequency.

If a system is subjected to a kinematic excitation $r(t)$, the generalized force $S(t, x) = s(t)Y(x)$ corresponds to the inertia forces generated by the acceleration $s(t) = v^{\cdot}(t) = r^{\cdot\cdot}(t)$. This implies that both $v(t)$ and $r(t)$ must be processes of limited variance. The spectrum of the process $s(t)$ then has the form (1.15) with $k = 2$, and it also has a cut-off frequency ω^*. If this frequency is higher than the natural frequencies of the system, $s(t)$ can be considered as a high-frequency process. It follows from (1.15) in particular, that the system cannot be stabilized by a parametric perturbation $s(t)$ of the "white noise" type. The introduction of a small parameter for random perturbations is discussed in Section 2 using an example.

2. THE STABILITY OF THE EQUILIBRIUM POSITIONS OF A PENDULUM ON AN ELASTIC SUSPENSION

As an example, we will investigate the change in the equilibrium positions of a pendulum OC on an elastic suspension DC (Fig. 1).

The pendulum moves in a horizontal plane, i.e. the effect of gravity can be ignored. The centre of mass of the pendulum is at the point C. The axis of rotation O of the pendulum is located under the fixing point D of the suspension; without loss of generality we will simplify the calculations by putting $OD = OC = l$. The kinetic energy of the pendulum is $T = ml^2(\theta^{\cdot})^2/2$, that is, $A(\theta) = ml^2$, $a(\theta) = 1/(ml^2)$, where m is the mass of the pendulum, and θ is the angle between the pendulum and the axis OD. The suspension point O oscillates with acceleration $v^{\cdot}(t)$ along the vertical axis. Following D'Alembert's

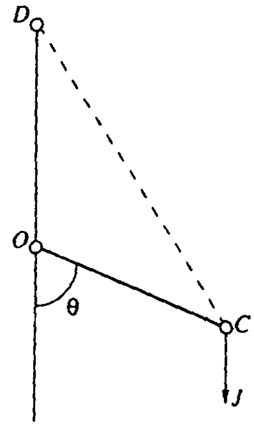

Fig. 1

principle, we change kinematic excitation to the inertia forces, and consider the relative motion as oscillations of the pendulum with a fixed axis acted upon by the inertia force $J = m\, v^{\cdot}(t)$ applied at the centre of mass (Fig. 1). The potential energy of the elastic suspension is written in the form

$$\Pi(\theta) = 2ml^2k^2[\cos(\theta/2) - \lambda]^2, \quad k^2 = c/m, \quad \lambda = l_0 2l < 1;$$

where l_0 is the length of the non-deformed suspension. The generalized force corresponds to the moment of the inertia force, i.e.

$$L(t,\theta) = mv^{\cdot}(t)l\sin\theta$$

If the oscillation of the base is harmonic, $r(t) = \alpha \sin\omega$, the small parameter ε is introduced taking into account the relations between the parameters of the system $\alpha \ll l$, $k \ll \omega$. We now introduce a small parameter in the case of random oscillations of the suspension when the amplitude and frequency of the perturbation cannot be clearly distinguished. For simplicity, we assume $v^{\cdot}(t)$ to be a process with bounded variance, $\langle [v^{\cdot}(t)]^2 \rangle = D^2$. Put $v^{\cdot}(t) = \sigma\omega w(t)$, where $\sigma^2 = \langle v^2(t) \rangle$, and $w(t)$ is a dimensionless acceleration such that $\langle w^2(t) \rangle = 1$. Then $\langle [v^{\cdot}(t)]^2 \rangle = (\sigma\omega)^2$, $\omega = D/\sigma$. Bearing in mind the relations between the system parameters, we let $k/\omega = \varepsilon$, $\sigma/l\omega = \varepsilon\rho$.

With these assumptions we can write.

$$\Pi(\theta) = \varepsilon^2\Phi(\theta) = 2\varepsilon ml^2\omega^2[\cos(\theta/2) - \lambda]^2$$

$$L(t,\theta) = \varepsilon S(t,\theta) = \varepsilon ml^2\omega^2\rho w(t)\sin\theta$$

As a result of transformations, taking (1.9) into account we obtain

$$F(\theta) = \frac{1}{2}ml^2\omega^2\rho^2\sin^2\theta$$

Then

$$U(\theta) = \Phi(\theta) + F(\theta) = ml^2\omega^2\left[2(\cos\frac{\theta}{2} - \lambda)^2 + \frac{1}{2}\rho^2\sin^2\theta\right] \tag{2.1}$$

Extrema of the effective potential (2.1) can be found from the equation

$$R(\theta) = -2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \lambda\right) + \rho^2\sin\theta\cos\theta = 0 \tag{2.2}$$

(a positive constant coefficient is omitted). Two equilibrium positions exist in the unperturbed system ($\rho = 0$), defined by the condition

$$K(\theta) = -2k^2 \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \lambda\right) = 0; \quad \theta_0 = 0, \quad \theta_1 = 2\arccos\lambda \tag{2.3}$$

It can be shown that the equilibrium position $\theta_0 = 0$ is unstable, whereas θ_1 is stable. To determine the perturbed equilibrium positions, we transform Eq. (2.2) to the form

$$R(\theta) = 2\sin\frac{\theta}{2}[-z + \lambda + \rho^2 z(2z^2 - 1)] = 0, \quad z(\theta) = \cos\frac{\theta}{2} \in [0, 1] \tag{2.4}$$

The system retains the equilibrium position $\theta_0 - 0$. It can be verified that this position is stable if $\rho^2 > 1$. Additional equilibrium positions can be found as roots of the cubic equation (Fig. 2)

$$f(z) = 2\rho^2 z^3 - (1 + \rho^2)z = -\lambda, \quad 0 \leqslant z \leqslant 1 \tag{2.5}$$

Elementary analysis shows that when the condition

$$\frac{2}{3\sqrt{6}}(1 + \mu^2)^{3/2} > \mu^2\lambda, \quad \mu = \frac{1}{\rho} \tag{2.6}$$

is satisfied Eq. (2.5) has three real roots, two of which are positive. In Fig. 2 these roots are defined as points of intersection of the straight line *1* with the graph of function (2.5). If this condition is not satisfied, Eq. (2.5) has a unique real negative root, defined as the point of intersection of the straight line 2 with the graph of function (2.5). This root is not taken into account when analysing the stability.

Condition (2.6) is always satisfied if $\mu = 1/\rho < 1$ is sufficiently small and $\lambda < 1$. The roots of Eq. (2.6) and the corresponding equilibrium positions have the form

$$z_1 \approx \frac{\sqrt{2}}{2} + \mu^2\left(\frac{\sqrt{2}}{4} - \lambda\right), \quad \theta_1 \approx \frac{\pi}{2} - \mu^2\left(1 - \frac{4}{\sqrt{2}}\lambda\right) \tag{2.7}$$

$$z_2 \approx \mu^2\lambda, \quad \theta_2 \approx \pi - 2\mu^2\lambda$$

Investigating the stability of the equilibrium positions we obtain that the positions $\theta_0 = 0$ and $\theta_2 \approx \pi - 2\mu^2\lambda$ are stable, and the position θ_1 is unstable when $\mu < 1$, $\rho > 1$.

Condition (2.6) holds for fairly small $\rho \ll 1$, but in this case the maximum root $z_1 > 1$. Hence two equilibrium positions exist, the unstable position $\theta_0 = 0$ and the stable position θ_2 corresponding to the minimum positive root $z_2 \approx \lambda\,(1 + \rho^2)$, that is $\theta_2 \approx (\arccos\lambda + \sqrt{2}\rho/4)$. This implies that weak fast perturbations when $\rho \ll 1$ result in only a slight displacement of the equilibrium position. If $\rho \gg 1$, the perturbation is fairly large and new stable equilibrium positions appear.

3. THE MOTION OF A SPHERE SUBJECTED TO A HIGH-FREQUENCY LOAD

We will now investigate, as an example of a system with cyclic coordinates, a sphere which rolls without slipping over a smooth horizontal plane.

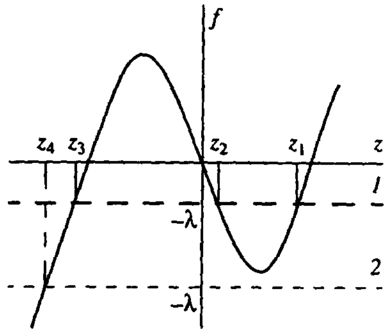

Fig. 2

The centre of gravity D of the sphere with the coordinates x_D and y_D lies on the axis of dynamic symmetry but does not coincide with the geometric centre O of the sphere, and $OD = d \neq 0$. The coordinates x_D and y_D and the Euler angles φ, ψ, θ are chosen as independent coordinates. Here θ is the angle of nutation between the vertical axis z and the axis of dynamic symmetry ζ, ψ is the angle of precession and φ is angle of pure rotation around the ζ axis (Fig. 3). We will investigate the motion of the sphere under the inertial excitation $J = h\omega^2 \sin\omega t$ applied at the centre of mass, under the condition that the sphere executes continuous motion along the plane. The kinetic energy of the body is given by [15]

$$2T = (K + Md^2\sin^2\theta)\theta^{\cdot 2} + K\psi^{\cdot 2}\sin^2\theta + C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2 + Mv_D^2 \tag{3.1}$$

where M is the mass of the body, C is the moment of inertia about the ζ axis, the central moments of inertia about the two other principal axes are equal to K, and the velocity of the centre of gravity is $v_D^2 = x_D^{\cdot 2} + y_D^{\cdot 2}$. The potential energy of the sphere is

$$\Pi(\theta) = -Mgd\cos\theta \tag{3.2}$$

the force J generates a torque around the axis of nutation (Fig. 3)

$$Y(\theta) = -Jd\sin\theta = -hd\omega^2\sin\theta\sin\omega t \tag{3.3}$$

It follows from (3.1)–(3.3) that x_D, y_D, φ, ψ are cyclic coordinates and θ is a positional coordinate. The corresponding momenta take the form

$$\begin{aligned} &p_\theta = A(\theta)\theta^{\cdot}, \quad A(\theta) = K + Md^2\sin^2\theta \\ &p_x = Mx_D^{\cdot} = m, \quad p_y = My_D^{\cdot} = n \\ &p_\varphi = C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta) = G, \quad p_\psi = K\psi^{\cdot}\sin^2\theta + G\cos\theta = D \end{aligned} \tag{3.4}$$

where m, n, G and C are constants. From (3.1), (3.2) and (3.4) we obtain the Routh potential in the form [15]

$$\Pi^*(\theta) = \frac{1}{2K\sin^2\theta}(D - G\cos\theta)^2 - Mgd\cos\theta + \frac{1}{2M}(m^2 + n^2) \tag{3.5}$$

We introduce a small parameter ε. Let the natural frequency of rotational oscillations under gravity be small compared to the frequency of external excitation, that is $g/d\omega^2 = \omega^2$. Making the usual assumptions regarding the relations between the system parameters, we define

$$\frac{h}{Md} = \varepsilon\mu, \quad \frac{G}{K\omega} = \varepsilon\gamma, \quad \frac{D}{K\omega} = \varepsilon\delta, \quad \frac{Md^2}{K} = \beta$$

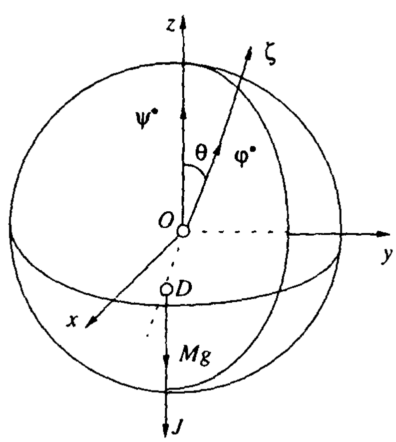

Fig. 3

Relations (1.8) and (3.3)–(3.5) then yield.

$$Y(\theta) = \varepsilon S(t,\theta), \qquad S(t,\theta) = -\omega^2 K\mu\beta \sin\omega t \sin\theta$$

which gives

$$F(\theta) = \frac{1}{2}\omega^2 K\mu^2 \frac{\beta^2\sin^2\theta}{1+\beta\sin^2\theta}, \qquad \Pi^*(\theta) = \varepsilon^2\Phi^*(\theta)$$

i.e.

$$\Phi^*(\theta) = \frac{1}{2}\omega^2 K\left[\frac{(\delta-\gamma u)^2}{1-u^2} - 2\beta u\right], \qquad u = \cos\theta \tag{3.6}$$

(the constant component in $\Phi^*(\theta)$ is dropped).

Consider in detail the non-singular case $\sin\theta \neq 0$, $\delta \neq \gamma$ when $\delta\gamma > 0$. From Eqs (1.11) and the expression for the effective potential $U = F + \Phi^*$ we obtain an equation for determining the equilibrium positions.

$$R(\theta) = \frac{1}{2}\omega^2 K \sin\theta\left\{\frac{(\delta-\gamma u)(\gamma-\delta u)}{(1-u^2)^2} + 2\beta + 2\beta^2\mu^2\frac{u}{[1+\beta(1-u^2)]^2}\right\} = 0 \tag{3.7}$$

The roots of this equation can be obtained graphically as the points of intersection of the curves

$$f(u) = \frac{(\delta-\gamma u)(\gamma-\delta u)}{(1-u^2)^2}, \qquad g(u) = -2\beta\left\{1+\beta\mu^2\frac{u}{[1+\beta(1-u^2)]^2}\right\} \tag{3.8}$$

If δ and γ have the same sign, then, when $\beta > 0$, the unperturbed system has a single stable equilibrium position $u^* > 0$, $\theta^* = \arccos u^*$ corresponding to the centre of gravity below the geometrical centre of the sphere [15] (Fig. 4).

The angle of inclination of the axis OD decreases when μ is fairly small, but the equilibrium position remains unique and stable. As the parameter μ increases, three points of intersection appear. These points correspond to a lower ($\mu_1 > 0$), upper ($\mu_2 > 0$), and intermediate ($u_2 < 0$, $\theta_2 = \arccos u_2 > \pi/2$) equilibrium positions.

We will obtain an approximate value of u_2 and estimate the stability of the corresponding equilibrium position. Suppose $\theta_2 = \pi/2 + \eta$, $\eta \ll 1$; we then find $u_2 \approx -\eta$ and neglect the terms of order η^2 in Eq. (3.7). The root u_2 can then be found from the linearized equation

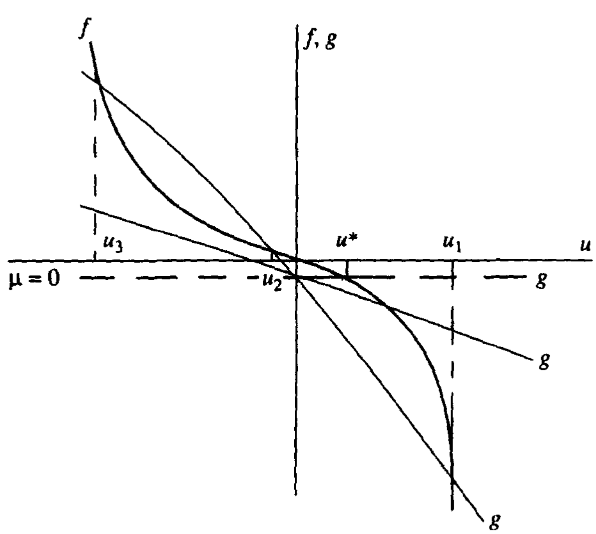

Fig. 4

$$\delta\gamma + 2\beta + (\rho^2 - \gamma^2 - \delta^2)u = 0; \quad \rho^2 = 2\beta^2\mu^2(1+\beta)^{-2} \tag{3.9}$$

The root $u_2 = (\gamma^2 + \delta^2 - \rho^2)^{-1}(\delta\gamma + 2\beta) < 0$ exists when $\delta\gamma > 0$ provided $\rho^2 > \gamma^2 + \delta^2$, i.e. the perturbation intensity is high.

We will investigate the stability of the position θ_2. When $\theta_2 = \pi/2 + \eta$, $\eta \ll 1$ we have

$$dR/d\theta = -\omega^2 K(\rho^2 - \gamma^2 - \delta^2)/2 < 0$$

i.e. the intermediate position is unstable, and, in turn, the lower position $\theta_1 < \pi/2$ and the upper position $\theta_3 > \theta_2 > \pi/2$ are stable. This indicates that high-frequency excitation can sustain stable motion in which the centre of gravity of the sphere lies above its geometrical centre. If the perturbations are weak, $\rho^2 < \gamma^2 + \delta^2$, only the lower equilibrium position $\theta^* < \pi/2$ exists.

We will now investigate the existence of regular precession in the motion of the sphere. From (3.4) we obtain $\delta = \gamma u + \psi^{\cdot}(1 - u^2)$. On reduction Eq. (3.7) takes the form

$$-u[\psi^{\cdot 2}(1-c) + 2\beta^2\mu^2(1+\beta-\beta u^2)^{-2}] = \beta + c\varphi^{\cdot}\psi^{\cdot}; \qquad c = C/K \tag{3.10}$$

If $|u| < 1$, equality (3.10) can be regarded as the condition for regular precession to exist. A solution $|u| < 1$ exists if

$$|\psi^{\cdot 2}(1-c) + 2\beta^2\mu^2(1+\beta)^{-2}| > |\beta + c\varphi^{\cdot}\psi^{\cdot}| \tag{3.11}$$

Regular precession in the unperturbed system exists if

$$\psi^{\cdot 2}|1-c| > |\beta + c\varphi^{\cdot}\psi^{\cdot}| \tag{3.12}$$

High-frequency perturbation weakens the condition for regular precession to exist, since it is possible for the following conditions to be satisfied

$$\psi^{\cdot 2}|1-c| < \beta + c\varphi^{\cdot}\psi^{\cdot} < |\psi^{\cdot 2}(1-c) + 2\beta^2\mu^2(1+\beta)^{-2}| \tag{3.13}$$

Hence high frequency perturbation can produce not only new equilibrium positions, but also stable steady motions not found in the unperturbed system.

A similar analysis can be carried out for the case when $\delta\gamma < 0$.

We will consider some special cases.

(1) $\theta = 0, u = 1, \delta = \gamma$. Calculating the derivative of $R(\theta)$ when $\theta = 0$ we obtain $dR/d\theta > 0$ for all values of the parameters, including $\mu = 0$ (an unperturbed system).

(2) $\theta = \pi, u = -1, \delta = -\gamma$. Calculating $dR/d\theta$ when $\theta = \pi$ we obtain

$$dR/d\theta > 0, \qquad \gamma^2/4 > 2\beta(1 - \beta\mu^2) \tag{3.14}$$

From inequalities (3.14) it follows that the upper equilibrium position in the unperturbed system is unstable if $\gamma^2/4 < 2\beta$. This means that high-frequency perturbation can stabilize an unstable equilibrium position.

4. INCREASE IN THE CRITICAL LOAD DUE TO FAST PULSATION OF THE TRACKING FORCE

As an example of a system acted upon by non-potential rapidly varying forces, we will consider a double pendulum driven by a tracking force. The pendulum consists of two similar weightless rods of length l and mass m concentrated at the ends of the rods, joined by elastic hinges of stiffness c. The pendulum is set up on a fixed base. The tracking force F tracks the position of the upper rod (Fig. 5).

The loss of stability of a system on a fixed base has been studied in detail [18]. We know that the achievement of the critical level of loading gives rise to instability. The critical load increases in the case of quasi-periodic vibrations of the base [5].

We will investigate the effect of pulsation of the load on the system stability. Suppose the tracking force has the form $F = F_0 + F_\varepsilon(\tau)$, where F_0 is a constant component and $F_\varepsilon(\tau)$ is a rapidly varying alternating component satisfying the conditions of Section 1. Motion occurs in a horizontal plane and the effect of gravity and dissipative forces is neglected. A detailed analysis of the equilibrium positions

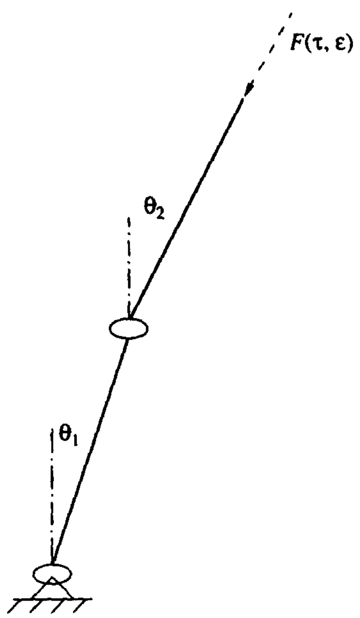

Fig. 5

and their stability is quite complicated and requires a numerical analysis. We will therefore restrict our consideration to determining the critical load which loads to a loss of elastic stability.

The linearized equations of slow motion in the neighbourhood of the equilibrium position $\theta_1 = \theta_2 = 0$ have the form

$$\begin{aligned} &2ml^2\theta_1'' + ml^2\theta_2'' + (2c - Fl)\theta_1 - (c - Fl)\theta_2 = 0 \\ &ml^2\theta_1'' + ml^2\theta_2'' - c\theta_1 + c\theta_2 = 0 \end{aligned} \tag{4.1}$$

where the prime denotes the derivative with respect to the slow time $\tau = \varepsilon t$, where ε is a small parameter defined in the standard way. Changing to the fast variable t system (4.1) can be reduced to the form (1.1)

$$\begin{aligned} &\ddot{\theta}_1 + [\varepsilon^2(3k^2 - f_0) - \varepsilon w(t)]\theta_1 + [\varepsilon^2(f_0 - 2k^2) + \varepsilon w(t)]\theta_2 = 0 \\ &\ddot{\theta}_2 + [\varepsilon^2(f_0 - 4k^2) + \varepsilon w(t)]\theta_1 + [\varepsilon^2(3k^2 - f_0) - \varepsilon w(t)]\theta_2 = 0 \end{aligned} \tag{4.2}$$

where

$$\varepsilon^2 k^2 = c/(ml^2), \quad \varepsilon^2 f_0 = F_0/(ml), \quad \varepsilon w(t) = F_\varepsilon/(ml), \quad w(t) = \dot{\upsilon}(t)$$

The characteristic Eq. (1.14) $\varepsilon^2k^2 = c/(ml^2)$, $\varepsilon^2 f_0 = F_0/(ml)$, $\varepsilon w(t) = F_\varepsilon(ml)$, $w(t) = \dot{\upsilon}(t)$, corresponding to system (4.2), takes the form

$$[p^2 + (3k^2 - \xi)]^2 - (\xi - 2k^2)(\xi - 4k^2) = 0; \quad \xi = f_0 - 2\sigma^2, \quad \sigma^2 = \langle \upsilon^2(t) \rangle \tag{4.3}$$

Analysing the rods of Eq. (4.3) we obtain the following conditions [18]:
stabilization

$$p^2 < 0, \quad f_0 < 2(k^2 + \sigma^2)$$

dynamic instability (flutter)

$$\operatorname{Im} p^2 \neq 0, \quad 2(k^2 + \sigma^2) < f_0 < 4k^2 + 2\sigma^2$$

static instability (divergence)

$$p^2 > 0, \quad f_0 > 4k^2 + 2\sigma^2$$

Thus the critical load satisfies the inequality

$$f_0 > 2(k^2 + \sigma^2)$$

It is easy to see that the critical value of the tracking force increases compared with the unperturbed system ($\sigma = 0$).

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